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# Bounded Distributed Average Tracking of General Linear Multi-Agent Systems Based on Sample-Data

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## ABSTRACT

***A distributed average tracking protocol based on sampled data is proposed for general linear multi-agent systems, which can guarantee that general linear multi-agent systems track multiple time-varying signals with general linear dynamics in a distributed manner. Based on the relative information among the neighboring agents, the states of each agent will converge to the average of the multiple reference signals if the network topology is undirected and connected. Algebraic graph theory, matrix theory and the discrete Lyapunov method are employed to theoretically prove the validity of the proposed protocol, and derive the sufficient conditions to ensure the system stability. The analysis results show that the range of the sampling period is related to the system matrix and that the steady-state error of the system is related to the control gain and the sampling period. Finally, three simulation examples are presented to validate the theoretical results.***

**Keywords:** bounded distributed average tracking, general linear multi-agent systems, sampled-data, time-varying reference signals

## 1. INTRODUCTION

Recently, distributed coordination control of multi-agent systems ([1-3]) has received more and more attention because of its potential applications in formation control of unmanned air vehicles, coverage control of wireless sensor networks, rendezvous control of multi-robotic systems and so on. The distributed average tracking (DAT) problem is considered as an extension of the consensus problem ([4]) and cooperative tracking problem ([5]), and its aim is to ensure a group of agents to track the average of multiple reference signals via local information exchange. The DAT problem has its unique significance and challenges, since the average value to be tracked is a function of multiple time-varying signals, which is unknown to all agents. The solution to this problem is very significant in many practical applications, such as information fusion of multiple sensors ([6, 7]), coordination control of multi-robotic systems ([8]), the algorithm design of distributed optimization ([9]) and so on.

Some researchers investigate the DAT problem of first-order multi-agent systems ([10, 11]). A distributed discontinuous control protocol is proposed for first-order multi-agent systems with fixed and switching topologies and the stability of the system is analyzed based on the non-smooth analysis method ([10]). The robustness of the system to initialization errors and time delays are further considered in [11]. The DAT problem of second-order multi-agent systems are studied ([12, 13]). A sliding-mode surface is designed to make trajectories on the surface achieve DAT ([12]). The DAT problem of a group of physical double-integrator agents under an undirected communication topology is addressed, where the requirement on velocity measurements is reduced ([13]). The DAT problem of multiple signals with linear dynamics is also studied ([14, 15]). Zhao et al. propose a pair of discontinuous algorithms with static and adaptive coupling strengths ([14]). Then, they redesign a new continuous algorithm via the boundary layer concept with clock synchronization devices ([15]). Two algorithms based on low-gain feedback and non-smooth feedback are proposed to deal with the DAT problem of multi-agent systems with input saturation ([16]). The inherent connection between the dynamic region-following formation control and the DAT problem is investigated in [17], which indicates that the dynamic region-following formation control algorithms based on DAT do not require the desired region to have a regular shape and can generate richer formation behavior. The DAT problem of networked Euler-Lagrange systems is studied in [18]. The extensions of the DAT problem in multi-agent coordination from three cases are made, and the experiments are carried out to verify the agents' task, which is to drive their center to track the average value of multiple reference signals through local interaction while maintaining connectivity and avoiding inter agent collision ([19]).

To the best of our knowledge, most ADT protocols in the existing literature assume that the communication between an agent and its neighborhood agents is continuous. Obviously, this assumption is inconsistent with practical applications. With the widespread application of digital sensors and controllers, control algorithms are usually implemented via the digital way. In many cases, although the system itself is a continuous process, only sampled data at discrete sampling instants can be available for the synthesis of control laws. Compared to the full state information transmission, the sampling state information transmission can effectively save the network bandwidth and communication costs. Considering the wide application of digital communication and control in distributed systems, the DAT problem of multi-agent systems based on sampled data is very significant. As far as we know, there is no open result about the DAT problem of multi-agent systems based on sampled data.

Motivated by the above observations, we consider the DAT problem of general linear multi-agent systems based on sampled data in this paper. Firstly, the state space model of a linear time-invariant continuous system with the continuous-time control protocol ([15]) is discretized by the ZOH technique, and a distributed control protocol based on sampled data is induced. Secondly, algebraic graph theory, matrix theory and the discrete Lyapunov method are employed to derive the sufficient conditions for the system to achieve DAT within an adjustable range.

The rest of this paper is organized as follows. In Section 2 some preliminaries are provided and the problem formulation is addressed. The stability of the DAT system is analyzed in Section 3. Numerical simulations are provided to illustrate the effectiveness of the theoretical results in Section 4. Finally, conclusions are presented in Section 5.

## 2. PRELIMINARIES AND PROBLEM FORMULATION

### 2.1 Some notations

The following notations will be used throughout this paper. $\mathbb{R}$, $\mathbb{R}^N$, and $\mathbb{R}^{N\times M}$ denote the set of real numbers, the $N$-dimensional Euclidean vector space and the set of $N\times M$ real matrices, respectively. $\mathbf{1}_N$ denotes the $N\times 1$ column vector of all ones and $I_N$ denotes the $N\times N$ identity matrix. For a vector $x=\left[x_1,x_2,\ldots,x_N\right]^T\in\mathbb{R}^N$, $\|x\|_p$ denotes the $p$-norm of $x$. For a matrix $Q\in\mathbb{R}^{N\times N}$, $Q^T$, $\|Q\|_2$, $\det(Q)$, $\lambda_{\max}(Q)$ and $\lambda_2(Q)$ denote its transpose, 2-norm and determinant, the maximum eigenvalues and the second smallest eigenvalues, respectively. $\operatorname{sgn}(\cdot)$ represents the signum function defined component-wise. The Kronecker product $A\otimes B$ of two matrices $A=\left[a_{ij}\right]\in\mathbb{R}^{n\times m}$ and $B=\left[b_{ij}\right]\in\mathbb{R}^{p\times q}$ is defined as $A\otimes B=\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$. The Kronecker product has the following properties ([20]): $c(A\otimes B)=cA\otimes B=A\otimes cB, c\in\mathbb{R}$; $(A+B)\otimes C=A\otimes C+B\otimes C$; $(A\otimes B)^T=A^T\otimes B^T$; $(A\otimes C)(B\otimes D)=AB\otimes CD$.

### 2.2 Algebraic graph theory

Graph $G\triangleq(V,E)$ is used to describe the communication topology among agents, where $V\triangleq\{1,\ldots,N\}$ is the node set and $E\subseteq V\times V$ is the edge set. An edge $(i,j)\in E$ means that nodes $i$ and $j$ can obtain information from each other, and they are neighbors of each other. In this paper, we only consider a simple graph, that is, $G$ does not contain self-loops. Graph $G$ is called to be undirected if for all $i,j\in V$, $(i,j)\in E \Leftrightarrow (j,i)\in E$. The adjacency matrix $A=\left[a_{ij}\right]\in\mathbb{R}^{N\times N}$ of graph $G$ is defined such that the edge weight $a_{ij}=1$ if $(i,j)\in E$ and $a_{ij}=0$ otherwise. The degree of node $i$ is defined as $\sigma_i\triangleq\sum_{j=1}^{N}a_{ij}$. The degree matrix is defined as $\Lambda\triangleq\operatorname{diag}\left(\left[\sigma_1,\ldots,\sigma_N\right]\right)\in\mathbb{R}^{N\times N}$. The

Laplacian matrix of $G$ is defined as $L \triangleq \Lambda - A \in \mathbb{R}^{N\times N}$. For an undirected graph $G$, $L$ is symmetric positive semidefinite. By arbitrarily assigning an orientation for the edges in $G$, let $D \triangleq \left[d_{ij}\right] \in \mathbb{R}^{N\times M}$ be the incidence matrix associated with $G$, where $d_{ij} = -1$ if the edge $e_j$ leaves from node $i$, $d_{ij} = 1$ if it enters node $i$, and $d_{ij} = 0$ otherwise. The Laplacian matrix $L$ is then given by $L = DD^T \in \mathbb{R}^{N\times N}$.

**Assumption 1**: The graph G is undirected and connected.

**Lemma 1** ([19]): Under Assumption 1, the Laplacian matrix $L$ has a simple zero eigenvalue and

$$0=\lambda_1(L) < \lambda_2(L) \le \cdots \le \lambda_{\max}(L)$$

where $\lambda_i(\cdot)$ denotes the $i$ th eigenvalue. For any vector $x \in \mathbb{R}^N$, $\lambda_2(L)x^T x \le x^T L x \le \lambda_{\max}(L)x^T x$.

**Lemma 2** ([21]): $D^T D$ and $L = DD^T$ have the same set of nonzero eigenvalues.

**Lemma 3** ([22]): For any vector $x \in \mathbb{R}^N$, $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$.

**Lemma 4** ([20]): Suppose that all eigenvalues of $A = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$ are $\lambda_1, \lambda_2, \cdots, \lambda_n$ and that all eigenvalues of $B = \left[b_{ij}\right] \in \mathbb{R}^{m\times m}$ are $\mu_1, \mu_2, \cdots, \mu_m$, then all eigenvalues of $A \otimes B$ are $\lambda_i \mu_j \left(i = 1,\cdots,n;\ j = 1,\cdots,m\right)$.

**Lemma 5** ([21]): For any vector $x \in \mathbb{R}^N$, if Assumption 1 holds, then

$$\operatorname{sgn}\left[D^T \varepsilon(t_k)\right]^T D^T D \operatorname{sgn}\left[D^T \varepsilon(t_k)\right] \le \lambda_{\max}(L) \left\|\operatorname{sgn}\left[D^T \varepsilon(t_k)\right]\right\|_2^2$$

### 2.3 Problem formulation

Consider multi-agent systems composed of $N$ agents with communication topology described by an undirected graph $G$. Suppose that there are $N$ agents with general linear dynamics given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i = 1,\ldots,N \tag{1}$$

where $A$ and $B$ are constant matrices with compatible dimensions, $x_i(t) \in \mathbb{R}^n$ is the state of agent $i$, and $u_i(t) \in \mathbb{R}^m$ is the control input of agent $i$ to be designed.

**Assumption 2**: The pair $(A, B)$ is stabilizable.

Suppose that each agent $i$ has a time-varying reference signal $r_i(t) \in \mathbb{R}^n$, $i = 1,\ldots,N$, which satisfies the following linear dynamics

$$\dot{r}_i(t) = Ar_i(t) + Bf_i(t), \quad i = 1,\ldots,N \tag{2}$$

where $r_i(t) \in \mathbb{R}^n$ represents the state of the $i$ th signal and $f_i(t) \in \mathbb{R}^m$ represents the reference input of the $i$ th signal. Here, we assume that $f_i(t)$ is measurable and bounded. i.e., $\|f_i(t)\|_\infty \le f_0$, where $f_0$ is a positive constant. Suppose that agent $i$ has access to $f_i(t)$ and $r_i(0)$, but does not have access to the other reference signals $r_j(t)$. We also assume that agent $i$ can obtain information from a subset of the other agents called its neighbors and denoted by $N_i$. Here we assume that $i \notin N_i$.

The DAT problem of system (1) has been investigated in [18, 19], but the proposed protocols are based on continuous communication between agents. Compared to the full-state information transmission, the sampling state information transmitting can save the network bandwidth and communication costs more effectively. In response to this, in this paper we propose the following sampled data-based protocol

$$u_i(t) = f_i(t_k) + \alpha \sum_{j\in N_i} \left[K(x_i(t_k) - x_j(t_k))\right] + \beta \sum_{j\in N_i} \operatorname{sgn}\left[K(x_i(t_k) - x_j(t_k))\right], \forall t \in \left[t_k, t_{k+1}\right), k = 0,1,\cdots \tag{3}$$

and initialize the states of all agents as

$$x_i(0) = r_i(0),\ i = 1,\ldots,N \tag{4}$$

where $t_k$ are the sampling instants satisfying $t_0 < t_1 < \cdots < t_k < \cdots$, the sampling period $h$ is a constant, i.e., $h = t_{k+1} - t_k$, $\alpha$ and $\beta$ are control gains to be determined, and $K$ is a feedback gain matrix to be determined.

It can be seen from protocol (3) that the control input of system (1) in the time interval $\left[t_k, t_{k+1}\right)$ is the same. Thus, we can get

$$x_i(t)=e^{A(t-t_k)}x_i(t_k)+\int_{t_k}^{t}e^{A(t_k-\tau)}Bu_i(\tau)d\tau,\forall t\in\left[t_k,t_{k+1}\right),k=0,1,\cdots,\ i=1,\cdots N \tag{5}$$

Then, the sampling control model of system (1) - (2) can be written as

$$x_i(t_{k+1})=A_0x_i(t_k)+B_0u_i(t_k) \tag{6}$$

$$r_i(t_{k+1})=A_0r_i(t_k)+B_0f_i(t_k) \tag{7}$$

where $A_0=e^{Ah}$ and $B_0=\left(\int_0^h e^{At}dt\right)B$.

From (3) and (6), we can obtain the closed-loop system in a matrix form as

$$x(t_{k+1})=\left(I_N\otimes A_0\right)x(t_k)+\left(I_N\otimes B_0\right)f(t_k)+\alpha\left(L\otimes B_0K\right)x(t_k)+\beta\left(D\otimes B\right)\text{sgn}\left[\left(D^T\otimes K\right)x(t_k)\right] \tag{8}$$

where $x(t_k)=\left[x_1^T(t_k),x_2^T(t_k),\ldots,x_N^T(t_k)\right]^T$, $f(t_k)=\left[f_1^T(t_k),f_1^T(t_k),\ldots,f_N^T(t_k)\right]^T$, $\text{sgn}(t_k)=\left[\text{sgn}_1(t_k),\text{sgn}_2(t_k),\ldots,\text{sgn}_N(t_k)\right]^T$.

System (8) will achieve DAT within a bounded range, if the final state of each agent $i$ satisfies

$$\lim_{k\to\infty}\left\|x_i(t_k)-\frac{1}{N}\sum_{j=1}^{N}r_j(t_k)\right\|_2\le C \tag{9}$$

where $C$ is a bounded positive constant.

It follows from (5) that system (1) under protocol (3) achieves the bounded DAT if and only if system (8) achieves the bounded DAT.

## 3. STABILITY ANALYSIS

In this section, we will theoretically analyze system (8) under the undirected and connected network topology. Before moving on, we need to provide and prove this following lemma.

**Lemma 6**: Under Assumption 1 and Assumption 2, if system (8) achieves consensus within a bounded range, i.e.,

$$\lim_{k\to\infty}\left\|x_i(t_k)-\frac{1}{N}\sum_{j=1}^{N}x_j(t_k)\right\|_2\le C,\ \text{for}\ i=1,\ldots,N$$

then

$$\lim_{k\to\infty}\left\|x_i(t_k)-\frac{1}{N}\sum_{j=1}^{N}r_j(t_k)\right\|_2\le C,\ \text{for}\ i=1,\ldots,N$$

**Proof**: It follows from Assumption 1 that

$$\alpha\sum_{i=1}^{N}\sum_{j\in N_i}\left[K(x_i(t_k)-x_j(t_k))\right]+\beta\sum_{i=1}^{N}\sum_{j\in N_i}\text{sgn}\left[K(x_i(t_k)-x_j(t_k))\right]=0 \tag{10}$$

From (6) and (7), we can obtain that

$$\sum_{i=1}^{N}x_i(t_{k+1})=A_0\sum_{i=1}^{N}x_i(t_k)+B_0\sum_{i=1}^{N}u_i(t_k) \tag{11}$$

$$\sum_{i=1}^{N}r_i(t_{k+1})=A_0\sum_{i=1}^{N}r_i(t_k)+B_0\sum_{i=1}^{N}f_i(t_k) \tag{12}$$

Combining (4), (11) and (12) leads to

$$\left|\sum_{i=1}^{N}x_i(t_{k+1})-\sum_{i=1}^{N}r_i(t_{k+1})\right|=A_0\left|\sum_{i=1}^{N}x_i(t_k)-\sum_{i=1}^{N}r_i(t_k)\right|=A_0\left|\sum_{i=1}^{N}x_i(0)-\sum_{i=1}^{N}r_i(0)\right|=0 \tag{13}$$

Therefore,

$$\sum_{i=1}^{N}x_i(t_k)=\sum_{i=1}^{N}r_i(t_k) \tag{14}$$

According to Assumption 1, one gets that if system (8) achieves consensus within a bounded range, i.e., $\lim_{k\to\infty}\left\|x_i(t_k)-\frac{1}{N}\sum_{j=1}^{N}x_j(t_k)\right\|_2\le C$, for $i=1,\ldots,N$, then, it follows from (14) that $\lim_{k\to\infty}\left\|x_i(t_k)-\frac{1}{N}\sum_{j=1}^{N}r_j(t_k)\right\|_2\le C$, for $i=1,\ldots,N$. The proof is completed.

Define $\varepsilon(t_k)=(M\otimes I)x(t_k)$, where $\varepsilon(t_k)=\left[\varepsilon_1^T(t_k),\varepsilon_2^T(t_k),\cdots,\varepsilon_N^T(t_k)\right]^T$ and $M=I_N-\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$. It can be seen that 0 is a simple eigenvalue of $M$ with $\mathbf{1}_N$ as the corresponding right eigenvector, and 1 is the other eigenvalue with multiplicity $N-1$. Then, it follows that $\varepsilon(t_k)=0$ if and only if $x_1(t_k)=x_2(t_k)=\cdots=x_N(t_k)$. In the following, we refer to $\varepsilon(t_k)$ as the consensus error. By noting that $M^2=M$, $M^T=M$, $D^TM=D^T$, $MD=D$, $ML=LM=L$, it is easy to obtain from (8) that the consensus error system satisfies

$$\varepsilon(t_{k+1})=(I_N\otimes A_0+\alpha L\otimes B_0K)\varepsilon(t_k)+(M\otimes B_0)f(t_k)+\beta(D\otimes B)\mathrm{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right] \tag{15}$$

According to Lemma 6, the DAT problem of system (1) is transformed into the stability problem of system (15). By analyzing the stability of system (15), we can get the following main results.

**Theorem 1**: Under Assumption 1 and Assumption 2, the state $x_i(t)$ of system (1) with protocol (3) will track the average of multiple reference signals $r_i(t)$ within a bounded range, if system (15) is exponentially stable within a bounded range, i.e.,

$$\lim_{k\to\infty}\|\varepsilon(t_k)\|_2\le C \tag{16}$$

and the designed parameters in protocol (3) satisfy

$$\begin{cases}K=-B_0^TA_0\\ \alpha>\dfrac{1}{\lambda_2(L)}\\ \beta\ge f_0(N-1)\end{cases} \tag{17}$$

where the range of the sampling period $h$ is determined by $\lambda_i(A_0-B_0K)<1$.

**Proof**: Construct a common Lyapunov function

$$V(\varepsilon(t_k))=\varepsilon(t_k)^T\varepsilon(t_k) \tag{18}$$

One gets

$$\Delta V(\varepsilon(t_k))=V(\varepsilon(t_{k+1}))-V(\varepsilon(t_k))=\varepsilon(t_{k+1})^T\varepsilon(t_{k+1})-\varepsilon(t_k)^T\varepsilon(t_k)=\Delta_1+\beta^2\Delta_2+\Delta_3+\beta\Delta_4+\Delta_5+\beta\Delta_6-\Delta_7 \tag{19}$$

where

$$\Delta_1=\varepsilon^T(t_k)(I_N\otimes A_0+\alpha L\otimes B_0K)^T(I_N\otimes A_0+\alpha L\otimes B_0K)\varepsilon(t_k)$$

$$\begin{aligned}\Delta_2&=\mathrm{sgn}^T\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right](D\otimes B_0)^T(D\otimes B_0)\mathrm{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\\&=\mathrm{sgn}^T\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\left(D^TD\otimes B_0^TB_0\right)\mathrm{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\end{aligned}$$

$$\Delta_3=f^T(t_k)(M\otimes B_0)^T(M\otimes B_0)f(t_k)=f^T(t_k)\left(M\otimes B_0^TB_0\right)f(t_k)$$

$$\begin{aligned}\Delta_4&=\varepsilon^T(t_k)(I_N\otimes A_0+\alpha L\otimes B_0K)^T(D\otimes B_0)\mathrm{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\\&\quad+\mathrm{sgn}^T\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right](D\otimes B_0)^T(I_N\otimes A_0+\alpha L\otimes B_0K)\varepsilon(t_k)\end{aligned}$$

$$\Delta_5=\varepsilon^T(t_k)(I_N\otimes A_0+\alpha L\otimes B_0K)^T(M\otimes B_0)f(t_k)+f^T(t_k)(M\otimes B_0)^T(I_N\otimes A_0+\alpha L\otimes B_0K)\varepsilon(t_k)$$

$$\Delta_6=\mathrm{sgn}^T\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right](D\otimes B_0)^T(M\otimes B_0)f(t_k)+f^T(t_k)(M\otimes B_0)^T(D\otimes B_0)\mathrm{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]$$

$$\Delta_7=\varepsilon(t_k)^T\varepsilon(t_k)$$

Because of the fact that $\delta^T\mathrm{sgn}(\delta)=\|\delta\|_1$, one has

$$\begin{aligned}\Delta_4 &= 2\varepsilon^T(t_k)\left(I_N\otimes A_0+\alpha L\otimes B_0K\right)^T\left(D\otimes B_0\right)\operatorname{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\\ &=2\left[\left(D^T\otimes B_0{}^TA_0\right)\varepsilon(t_k)\right]^T\operatorname{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\\ &\quad+2\alpha\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]^T\left(D^TD\otimes B_0{}^TB_0\right)\operatorname{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\\ &\le -2\left\|\left(D^T\otimes B_0{}^TA_0\right)\varepsilon(t_k)\right\|_1+2\alpha\lambda_{\max}\left(D^TD\otimes B_0{}^TB_0\right)\left\|\left(D^T\otimes B_0{}^TA_0\right)\varepsilon(t_k)\right\|_1\\ &=-2\left[1-\alpha\lambda_{\max}\left(D^TD\otimes B_0{}^TB_0\right)\right]\left\|\left(D^T\otimes B_0{}^TA_0\right)\varepsilon(t_k)\right\|_1\\ &=-\left[1-\alpha\lambda_{\max}\left(D^TD\otimes B_0{}^TB_0\right)\right]\sum_{i=1}^{N}\sum_{j\in N_i}\left\|B_0{}^TA_0\left(\varepsilon_i(t_k)-\varepsilon_j(t_k)\right)\right\|_1\end{aligned} \tag{20}$$

Similarly, one gets

$$\begin{aligned}\Delta_5 &= 2\varepsilon^T(t_k)\left(I_N\otimes A_0{}^T\right)\left(M\otimes B_0\right)f(t_k)+2\alpha\varepsilon^T(t_k)\left(L\otimes K^TB_0{}^T\right)\left(M\otimes B_0\right)f(t_k)\\ &=2\varepsilon^T(t_k)\left(M\otimes B_0{}^TA_0\right)^Tf(t_k)+2\alpha\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]^T\left(D^T\otimes B_0{}^TB_0\right)f(t_k)\end{aligned} \tag{21}$$

With the aid of $\|f_i(t)\|_2\le f_0$ and the Hölder's Inequality, we have

$$\begin{aligned}&\varepsilon^T(t_k)\left(M\otimes B_0{}^TA_0\right)^Tf(t_k)\\ &\le\left\|\left(M\otimes B_0{}^TA_0\right)\varepsilon^T(t_k)\right\|_1\|f(t_k)\|_\infty=\frac{f_0}{N}\sum_{i=1}^{N}\sum_{j\in N_i}\left\|B_0{}^TA_0\left(\varepsilon_i(t_k)-\varepsilon_j(t_k)\right)\right\|_1\\ &\le\frac{f_0}{N}\sum_{i=1}^{N}\max_i\left\{\sum_{j=1,j\ne i}^{N}\left\|B_0{}^TA_0\left(\varepsilon_i(t_k)-\varepsilon_j(t_k)\right)\right\|_1\right\}=f_0\max_i\left\{\sum_{j=1,j\ne i}^{N}\left\|B_0{}^TA_0\left(\varepsilon_i(t_k)-\varepsilon_j(t_k)\right)\right\|_1\right\}\\ &\le\frac{f_0}{2}(N-1)\sum_{i=1}^{N}\sum_{j\in N_i}\left\|B_0{}^TA_0\left(\varepsilon_i(t_k)-\varepsilon_j(t_k)\right)\right\|_1\end{aligned} \tag{22}$$

and

$$\begin{aligned}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]^T\left(D^T\otimes B_0{}^TB_0\right)f(t_k)&\le\lambda_{\max}\left(D^T\otimes B_0{}^TB_0\right)\left\|\left(D^T\otimes K\right)\varepsilon(t_k)\right\|_1\|f(t_k)\|_\infty\\ &\le -f_0\lambda_{\max}\left(D^T\otimes B_0{}^TB_0\right)\sum_{i=1}^{N}\sum_{j\in N_i}\left\|B_0{}^TA_0\left(\varepsilon_i(t_k)-\varepsilon_j(t_k)\right)\right\|_1\end{aligned} \tag{23}$$

Combining (21) and (22) results in

$$\Delta_5\le f_0(N-1)\left[1-\alpha\frac{2}{N-1}\lambda_{\max}\left(D^T\otimes B_0{}^TB_0\right)\right]\sum_{i=1}^{N}\sum_{j\in N_i}\left\|B_0{}^TA_0\left(\varepsilon_i(t_k)-\varepsilon_j(t_k)\right)\right\|_1 \tag{24}$$

Let $\beta\Delta_4+\Delta_5\le 0$. From Lemma 2 and Lemma 4, we can get

$$\begin{cases}0<\alpha\lambda_{\max}\left(D^TD\otimes B_0{}^TB_0\right)<1\\ \beta>f_0(N-1)\end{cases} \tag{25}$$

We also have

$$\begin{aligned}\Delta_6&\le\operatorname{sgn}^T\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\left(D\otimes B_0\right)^T\left(D\otimes B_0\right)\operatorname{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]+f^T(t_k)\left(M\otimes B_0\right)^T\left(M\otimes B_0\right)f(t_k)\\ &=\Delta_2+\Delta_3\end{aligned} \tag{26}$$

Further, we have

$$\Delta V(\varepsilon(t_k))\le\left(\Delta_1-\Delta_7\right)+\left(\beta+1\right)\left(\beta\Delta_2+\Delta_3\right) \tag{27}$$

where

$$\begin{aligned}(\Delta_1-\Delta_7)&=\varepsilon^T(t_k)\left\{\left[I_N\otimes A_0+\alpha\left(L\otimes B_0K\right)\right]^T\left[I_N\otimes A_0+\alpha\left(L\otimes B_0K\right)\right]-I_N\otimes I_n\right\}\varepsilon(t_k)\\&\le\sum_{i=1}^{N}\varepsilon_i^T(t_k)\left[\left(A_0-\alpha\lambda_2(L)B_0B_0{}^TA_0\right)^T\left(A_0-\alpha\lambda_2(L)B_0B_0{}^TA_0\right)-I_n\right]\varepsilon_i(t_k)\end{aligned}\tag{28}$$

Let $\alpha>\frac{1}{\lambda_2(\mathrm{L})}$ and $Q=\left(A_0-B_0B_0{}^TA_0\right)^T\left(A_0-B_0B_0{}^TA_0\right)-I_n$. According to the discrete Lyapunov stability criterion ([23]), we can know that there must be such a negative definite matrix $Q$, if $\lambda_i\left(A_0-B_0B_0{}^TA_0\right)<1$, where $A_0$ and $B_0$ are the matrices of system (6), which contain the sampling period. So, we can get a range of the sampling period that guarantees the system stability.

From Lemma 2, Lemma 3 and Lemma 6, we have

$$\begin{aligned}\Delta_2&=\mathrm{sgn}^T\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\left(D^TD\otimes B_0{}^TB_0\right)\mathrm{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\\&\le\lambda_{\max}\left(D^TD\otimes B_0{}^TB_0\right)\left\|\mathrm{sgn}\left[\left(D^T\otimes K\right)\varepsilon(t_k)\right]\right\|_2^2\\&\le\frac{N(N-1)}{2}\lambda_{\max}\left(D^TD\otimes B_0{}^TB_0\right)\le\frac{N(N-1)}{2}\lambda_{\max}(L)\lambda_{\max}\left(B_0{}^TB_0\right)\end{aligned}\tag{29}$$

$$\Delta_3=f^T(t_k)\left(M\otimes B_0{}^TB_0\right)f(t_k)\le\lambda_{\max}\left(M\otimes B_0{}^TB_0\right)\|f(t_k)\|_2^2\le\lambda_{\max}\left(M\otimes B_0{}^TB_0\right)f_0^2\le f_0^2\lambda_{\max}\left(B_0{}^TB_0\right)\tag{30}$$

From (29) and (30), it follows that

$$(\beta+1)(\beta\Delta_2+\Delta_3)<\lambda_{\max}\left(B_0{}^TB_0\right)(\beta+1)\left[\frac{\beta\lambda_{\max}(L)}{2}N^2+f_0^2\right]\triangleq\varpi\tag{31}$$

where $\varpi>0$. From (26) and (31), we have

$$\Delta V(\varepsilon(t_k))=V(\varepsilon(t_{k+1}))-V(\varepsilon(t_k))\le-\gamma V(\varepsilon(t_k))+\varpi\tag{32}$$

where $\gamma=\lambda_{\min}(Q)>0$.

Furthermore, we have

$$V(\varepsilon(t_k))\le V(\varepsilon(0))\mathrm{e}^{-\gamma t_k}+\frac{\varpi}{\gamma}\left(1-\mathrm{e}^{-\gamma t_k}\right)\tag{33}$$

This further leads to

$$\lim_{k\to\infty}V(\varepsilon(t_k))\le\frac{\lambda_{\max}\left(B_0{}^TB_0\right)(\beta+1)\left[\frac{\beta\lambda_{\max}(L)}{2}N^2+f_0^2\right]}{\lambda_{\min}(Q)}\tag{34}$$

From (18) and (34) we can get a limited value of $C$

$$C\triangleq\sqrt{\frac{\lambda_{\max}\left(B_0{}^TB_0\right)(\beta+1)\left[\frac{\beta\lambda_{\max}(L)}{2}N^2+f_0^2\right]}{\lambda_{\min}(Q)}}>0$$

and

$$\lim_{k\to\infty}\|\varepsilon(t_k)\|_2\le C$$

Therefore, system (15) is exponentially stable within a bounded range and system (1) under protocol (3) can achieve the bounded DAT. The proof is finished.

**Remark 1**: Literature [24] pointed out that when the sampling period is enough small and the requirements for discretization accuracy are not very high, the discrete linear system model obtained by the ZOH method is similar to the discrete model obtained by the Euler's method. That is, if the sampling period is enough small, then $A_0\approx I+hA$ and $B_0\approx hB$. This conclusion has important significance in numerical simulations.

**Remark 2**: Under the condition that the reference input signal and the communication topology remain unchanged and the tracking system is stable, when the error of the system is controlled in a certain range, the steady-state error depends

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on the control gain and the sampling period. By using the control variable method, it can be concluded that when the control gain keeps invariant, the smaller the sampling period is, the smaller the steady-state error is; when the sampling period keeps invariant, the larger the control gain is, the greater the steady-state error is.

**Remark 3**: The DAT protocol proposed in this paper contains the sign function, which will cause chattering in the closed-loop system. From the analysis and simulation in [25], we can see that because of the existence of the sign function, the system will eventually converge alternately positive and negative in a certain region. This conclusion will be of great help to the following simulation results.

## 4. NUMERICAL SIMULATIONS

In this section, numerical simulation experiments are provided to illustrate the effectiveness of the proposed protocol (3). Consider multi-agent systems with four agents. The network topology among them is shown in Figure 1.

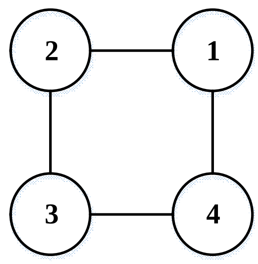
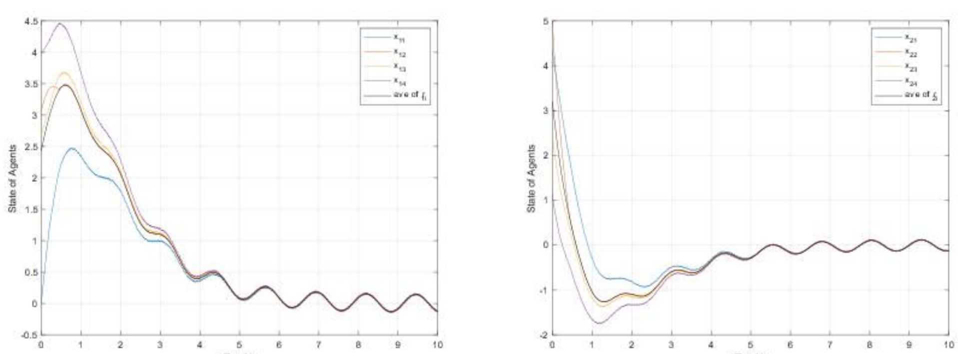



**Figure** 1 Network topology

The system matrices in (1) are chosen as $A=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$ and $B=\begin{bmatrix}1\\ 1\end{bmatrix}$. Generally, we assume that the weights of all edges of the undirected graph are 1 and the initial position states of the four agents are randomly generated in the range [0,5]. The input of the $i$ th reference signal is chosen as $f_i(t) = 0.25\times i\times \sin(5t)$, $i=1,2,3,4$. Through calculation, the maximum sampling period to keep the system stable is 0.366s. When the system is stable and the sampling period is small, the system matrices in (6)-(7) are $A_0 \approx I+hA=\begin{bmatrix}1 & h\\ -h & 1-2h\end{bmatrix}$ and $B_0 \approx hB=\begin{bmatrix}h\\ h\end{bmatrix}$, and the feedback gain matrix $K=-B_0^{\ T}A_0=\begin{bmatrix}h^2-h & h^2-h\end{bmatrix}$. Next, we give three different parameter values, from which we can see that the steady-state error of the system is related to the control gain and the sampling period. The numerical results are shown in Figures 2-4.




(a) The trajectories of $x_{i1}(t)$ (b) The trajectories of $x_{i2}(t)$

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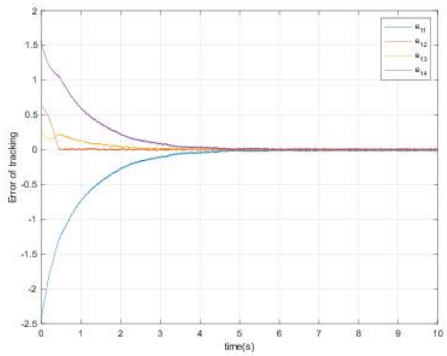
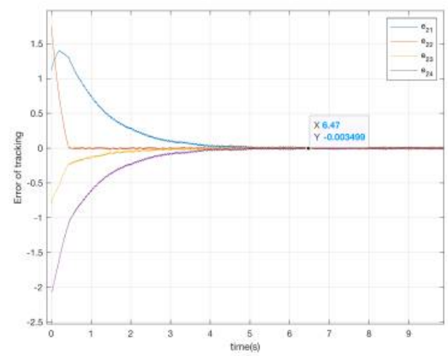
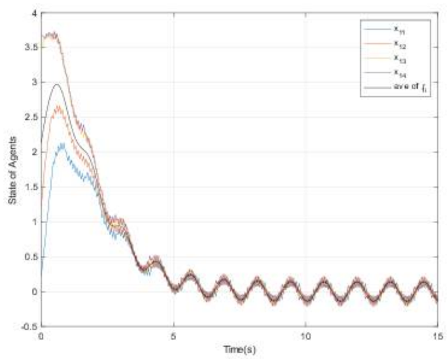



(c) The tracking errors $e_{i1}(t)$

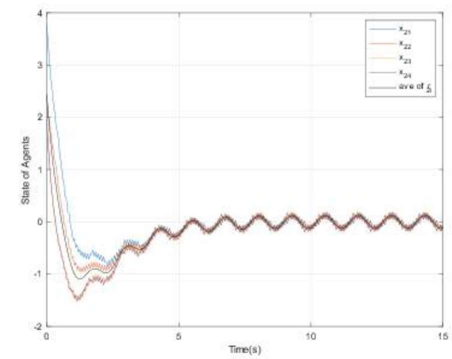



(d) The tracking errors $e_{i2}(t)$

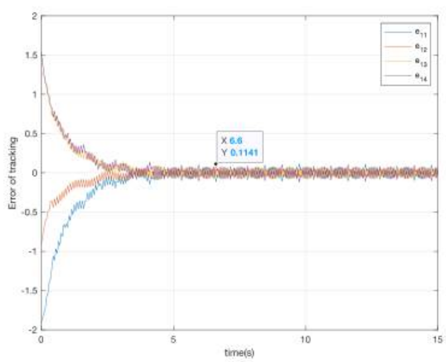
**Figure** 2 DAT of four agents under protocol (3) with $\alpha=1, \beta=4$ and $h=0.01$s



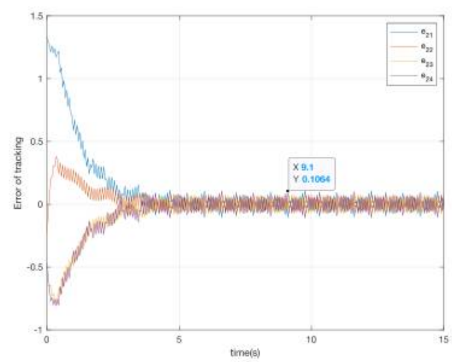

(a) The trajectories of $x_{i1}(t)$




(b) The trajectories of $x_{i2}(t)$

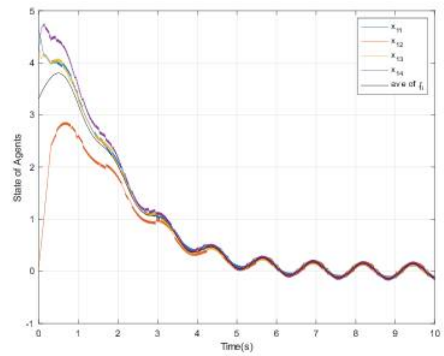



(c) The tracking errors $e_{i1}(t)$

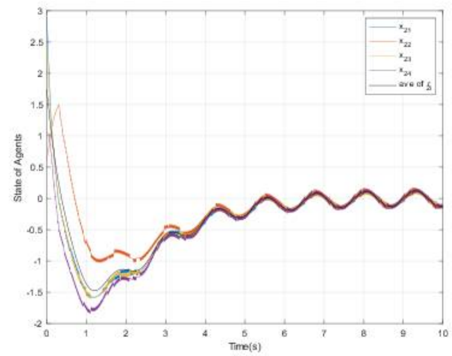



(d) The tracking errors $e_{i2}(t)$

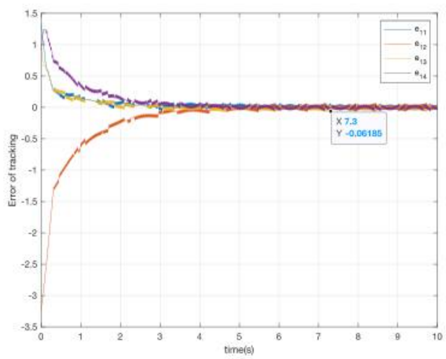
**Figure** 3 DAT of four agents under protocol (3) with $\alpha=1, \beta=4$ and $h=0.05$s



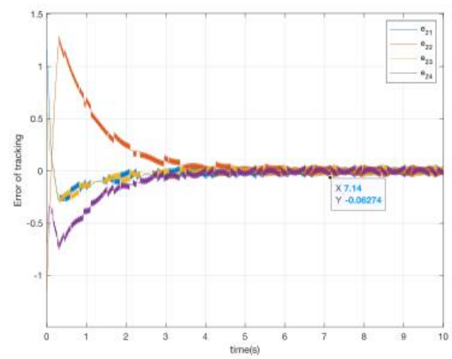

(a) The trajectories of $x_{i1}(t)$




(b) The trajectories of $x_{i2}(t)$




(c) The tracking errors $e_{i1}(t)$




(d) The tracking errors $e_{i2}(t)$

**Figure** 4 DAT of four agents under protocol (3) with $\alpha = 2, \beta = 6$ and $h = 0.01$s

## 5. CONCLUSIONS

This paper investigates the DAT problem of general linear multi-agent systems with multiple time-varying reference signals based on sampled data. We assume that the communication topology is undirected and connected and that the multiple time-varying reference signals have general linear dynamics. The ZOH method is used to transform the continuous-time model into a discrete-time model, and the DAT problem is transformed into a stability problem. Then, the discrete Lyapunov method is used for proving that the stability of the system can be controlled within a certain range by properly choosing the control gain and the sampling period, that is, each agent will eventually converge to the average of multiple reference signals within a bounded range. Finally, simulation examples verify the validity of the theoretical results.

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