DISTRIBUTED EVENT-TRIGGERED AVERAGE CONSENSUS OF MULTI-AGENT SYSTEMS

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ABSTRACT

This paper proposes a novel event-triggered consensus protocol for single-integrator multi-agent systems. The event detector is designed based on the local sampled-data, and information exchange is carried out only at corresponding event instants. The Lyapunov function method is employed to derive the sufficient conditions for average consensus. The bound of the inter-event time is ensured, and thus Zeno-behavior is avoided. Finally, the effectiveness of the proposed strategy is illustrated by numerical simulations.

KEYWORDS: average consensus, event-triggered, multi-agent systems, sampled-data

1. INTRODUCTION

Multi-agent systems are composed of multiple autonomous agents communicating with each other to actualize their cooperative control objectives such as swarming, flocking, formation, rendezvous, tracking and so on. In the past decade, synthesis and analysis of multi-agent systems have been extensively investigated by various researchers from multiple areas such as mathematics, physics, management science, and computer science. As a fundamental issue in distributed coordination control of multi-agent systems, consensus means that all agents using distributed control inputs called consensus protocols eventually reach a common value [1-5].

As the computer technology develops, an agent may be equipped with an embedded microprocessor to acquire information and update control input. However, since the power supply of an embedded microprocessor is constrained, it is urgent for researchers to devise a proper control algorithm for multi-agent systems such that the energy consumption can be reduced while the control performance is guaranteed. A traditional method is time-triggered control [6-8], in which the control execution and information transmission are in a periodic pattern. Although this method is effective for energy saving to some extent, the sampling period is often chosen according to the worst-case scenario, which may lead to unnecessary energy waste. Motivated by this conflict, the event-triggered control strategy has been subsequently proposed. In the event-triggered fashion, controller actuation only happens at some specific instants determined by the predefined event condition rather than continuously or in a periodic pattern. It was shown in Ref. [9] that compared with time-triggered control the event-triggered control is more preferred in practice since it eliminates unnecessary computation and communication.

Recently, event-triggered control has been extensively studied in interconnected systems [10-11]. In Ref. [12], the event-triggered mechanism is introduced into multi-agent systems, where both centralized and distributed event-triggered control strategies for each agent to determine control updates are proposed. Subsequently, the event-triggered control strategy is applied to the consensus problems of high-order multi-agent systems. In Ref. [13], the centralized event-triggered control strategy is used for consensus tracking of second-order multi-agent systems. There is a deficiency in Ref. [13] that all agents need to get the global information for event detection. This deficiency is remedied in Ref. [14]. Heterogeneous and general linear multi-agent systems via event-triggered control are investigated in Refs. [15-17]. Moreover, the event-triggered control strategies are proved to be applicable for multi-agent systems with more complicated cases, such as packet dropout [18], communication delays [19, 20], and noises [21]. In essence, the focus of event condition contains two factors: lower frequency of control updates and less information transmission among agents. In Ref. [19] a time-dependent event strategy is applied to the consensus problems for three kinds of multi-agent systems, where agents decide themselves when to communicate with their neighbors based only on local information, which quite reduces communication consumption. The homologous property is also preserved in Ref. [22], where continuous communication among agents is replaced with a discrete mode that information transmission only happens at corresponding event instants. However, control algorithms in Refs. [19, 22] contain implicit disadvantage that the control law for each agent is required to be updated at neighbors’ event instants. In order to solve this problem, a combinational measurement approach for event detection is investigated in Ref.
[23], where each agent equipped with the distributed strategy is only triggered at its own event instants. Nevertheless, the load of communication increases accordingly.

Based on the above analysis, most of the event conditions are continuous in the sense that agents need to monitor their own or neighbors’ states constantly, which not only requires delicate hardware, but also greatly augments the amount of communication and calculation. Motivated by this observation, in this paper we propose a novel distributed event-triggered consensus protocol for solving the average consensus problems of single-integrator multi-agent systems with a fixed, undirected, and connected network topology. The advantages of proposed distributed event-triggered consensus protocol are as follows: Firstly, different from centralized formulation, our approach only needs local information rather than global information to determine event instants. Secondly, the considered event-triggered mechanism is designed based on the sampled-data of each agent, in other words, agents only require periodic measurements of states for event detection, which is more practical for application. Similar work in Ref. [24] uses a different sampled-data event mechanism for solving the consensus problem with single integrator dynamics. However, we improve the results in Ref. [24] by further reducing the communication significantly. In the present communication policy, each agent determines when to exchange the information with its neighbors only based on local events rather than periodic pattern.

The remainder of this paper is organized as follows: Section 2 briefly introduces some concepts in algebraic graph theory and states the problem. Section 3 presents convergence analysis of the event-triggered consensus protocol. Numerical simulations are given to illustrate the effectiveness of our results in Section 4. Section 5 draws conclusions and discusses further research.

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 Algebraic graph theory and some notations

The interaction topology of multi-agent systems can be modeled by an undirected graph \( G = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \), which consists of a vertex set \( \mathcal{V} = \{\nu_i | i = 1, 2, \ldots, N\} \) representing \( N \) agents, an edge set \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) meaning the communication links between agents, and a weighted adjacency matrix \( \mathcal{A} = [a_{ij}] \). An edge between \( \nu_i \) and \( \nu_j \) is denoted by \( e_{ij} = (\nu_i, \nu_j) \). The adjacency element \( a_{ij} \) associated with the edge \( e_{ij} \) is positive, i.e., \( e_{ij} \in \mathcal{E} \leftrightarrow a_{ij} > 0 \). Moreover, we assume \( a_{ii} = 0 \) for all \( i \in \mathcal{I} \). For an undirected graph \( G \), the adjacency matrix \( \mathcal{A} \) is symmetric, i.e., \( a_{ij} = a_{ji} \). The set of neighbors of vertex \( \nu_i \) is denoted by \( \mathcal{N}_i = \{\nu_j | \nu_j \in \mathcal{V} : e_{ij} \in \mathcal{E} \} \). Correspondingly, the degree matrix \( \mathcal{D} \) of \( G \) is defined as \( \mathcal{D} = \text{diag}(d_1, d_2, \ldots, d_N) \) with \( d_i = \sum_{j=1}^{N} a_{ij} \). The Laplacian matrix of \( G \) is denoted by \( \mathcal{L} = \mathcal{D} - \mathcal{A} \). A path from \( \nu_i \) to \( \nu_j \) means a sequence of distinct edges \( (\nu_i, \nu_{i1}), (\nu_{i1}, \nu_{i2}), \ldots, (\nu_{i(k-1)}, \nu_j) \) in a graph. If there is a path between \( \nu_i \) and \( \nu_j \), then \( \nu_i \) and \( \nu_j \) are called connected. Graph \( G \) is connected if and only if there is a path between any two vertices. An important fact of \( \mathcal{L} \) is that all row sums are zero and thus \( \mathcal{L} \) has a right eigenvector \( 1_N \) associated with the zero eigenvalue, i.e., \( \mathcal{L}1_N = 0_N \), where \( 1_N \) and \( 0_N \) denote the \( N \)-dimensional column vectors with all ones and zeros, respectively. If graph \( G \) is connected, \( \mathcal{L} \) has one and only one zero eigenvalue and the remaining eigenvalues of \( \mathcal{L} \) are all positive.

The following notations will be used throughout this paper. \( \mathbb{R}^N \) means the \( N \)-dimensional Euclidean space. For a vector \( x \in \mathbb{R}^N \), its Euclidean norm is denoted as \( \|x\|_2 \). For a matrix \( A \in \mathbb{R}^{N \times N} \), the transpose matrix and the largest eigenvalue of \( A \) are denoted as \( A^T \) and \( \lambda_{\text{max}}(A) \), respectively. \( A > 0 \) or \( A < 0 \) represents that the matrix \( A \) is positive definite or negative definite, respectively.

2.2 Problem statement

Consider multi-agent systems composed of \( N \) agents. The dynamics of each agent is described by

\[
\dot{x}_i(t) = u_i(t), \quad i \in \{1, \ldots, N\},
\]

where \( x_i \in \mathbb{R} \) and \( u_i \in \mathbb{R} \) represent the state and control input of agent \( i \), respectively.

Our purpose is to design an event-triggered control strategy to solve the average consensus problem while reducing the energy consumption. Considering that the state measurement error plays a key role in event detection, we first define the following state measurement error for agent \( i \) as
where $t_i^k$ represents the $k$th event time of agent $i$, $h$ is the sample period for all agents synchronized physically by a clock. Combining the definition of the state measurement error in Equation (2) and

\[ z_i(t_i^k, \tau_i^k) = \sum_{j \in \mathcal{N}_i} (x_i(t_i^k) - x_j(t_j^k)) , \]

we propose the following event condition

\[ \left\| e_i(t_i^k + sh) \right\|_2^2 > \frac{m_i}{a} \left( \lambda_i - \frac{1}{a} - 2h \right) \left\| z_i(t_i^k, \tau_i^k) \right\|_2^2 , \]

where $m_i$ and $a$ are two positive constants to be determined latter.

It can be seen that the even condition (4) is designed based solely on the local state measurement error and discrete neighbors’ states. Each agent computes $e_i(t_i^k + sh)$ and checks the event condition at every sampling instant. Once the inequality (4) is satisfied, which means an event occurs, agent $i$ denotes event time by $t_i^k = t$, transmits its current state, i.e., $x_i(t_j^k) = x_j(t_j^k)$ to all of its neighbors, resets $e_i(t_i^k + sh)$ to zero, and actuates controller updates. Similarly, when agent $i$ receives $x_j(t_j^k)$ from its neighbor agent $j$ at corresponding times $t_j^k$, agent $i$ updates $z_i(t_i^k, \tau_i^k)$ by replacing $x_i(t_i^k)$ with $x_j(t_j^k)$. Therefore, $z_i(t_i^k, \tau_i^k)$ is only updated at all corresponding event times $t_i^k$ and $\tau_i^k$.

Following the above analysis, the event-triggered control law for each agent can be described by

\[ u_{i}(t) = -\sum_{j \in \mathcal{N}_i} (x_i(t_i^k) - x_j(t_j^k)), \quad t \leq t_i^k \in \mathcal{I}_{k+1}^i . \]

where $t_i^k$ is defined as

\[ t_i^k = \max \left\{ t \in \{ t_j^k, k = 0, 1, \ldots \}, t \leq t_i^k \right\} \]

which represents the last event time of agent $v_j$.

**Definition 1** The multi-agent systems (1) are said to achieve the average consensus, if

\[ \lim_{t \rightarrow \infty} x_i(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(0), \quad \forall t \in \{ 1, \ldots, N \} \]

holds for any initial conditions.

### 3. Convergence Analysis

In this section, we will theoretically show the effectiveness of the protocol (5) with the condition (4) to guarantee the systems (1) to achieve the average consensus. Before moving on, we need to provide the following lemma.

**Lemma 1** For multi-agent systems (1) with the consensus protocol (5) and the event condition (4). Assume that the network topology $G$ is fixed, undirected, and connected. Then for any agent $i$, the event times $t_i^k$ are integer multiple of $h$ and the inter-event time is lower bounded by $h$.

**Proof** Without loss of generality, assume that the considered systems contain two agents. From Figure 1, we suppose that $t = t_1^0 = kh$ is the first event time and then agent 1 will transmit its measurement to agent 2 at $kh$. Meanwhile, agent 2 will send its state back to agent 1 without any delay if its event condition is satisfied, otherwise no further action happens. Since the measurement error $e_1(t_1^0 + sh)$ remains zero, the agent 1 won’t be triggered again. Thus, any agent has to evaluate the event condition until the next sampling instant. Based on the above analysis, the event times for agent 1 can be defined iteratively by

\[ t_{k+1}^i = t_k^i + sh \inf \left\{ s h : \left\| e_i(t_i^k + sh) \right\|_2 > \frac{m_i}{a} \left( \frac{2}{\lambda_i} - \frac{1}{a} - 2h \right) \left\| z_i(t_i^k, \tau_i^k) \right\|_2 \right\} . \]

Furthermore, the interval between two events is larger than $h$, and the proof is completed.

![Figure 1 State update and communication process for two agents](image-url)
Remark 1 While the event condition (4) seems similar to that in Ref. [24], they are entirely different. In view of the threshold in Ref. [24], it contains a discrete variable \( z_i(t'+kh) \) given by \( z_i(t'+kh) = \sum_{j \in \mathcal{N}_i} (z_j(t' + kh) - z_i(t')) \). It is evident that \( z_i(t' + kh) \) should be updated at every sampling instant by the measurements of agent \( v_i \) and the received states of all neighbors. This approach will make the network consume immense resources for communication. Our work is devoted to reducing both computation and communication, so we preserve the property in Ref. [24] while further reducing the updates of \( z_i \). In our event threshold, agents notify its neighbors to update the event condition only at corresponding event times, in other words, the average communicating period in the present work is larger than the sampling period \( h \) in general.

In order to reduce clutter in the notations, we define \( \tilde{z}_i(t') = \sum_{j \in \mathcal{N}_i} (z_j(t') - z_i(t')) \). Combining Equation (1) and Equation (5), we can obtain the closed form of the systems (1) as follows
\[
\dot{x}_i(t) = - \sum_{j \in \mathcal{N}_i} (x_i(t_j') - x_j(t_j'))
\]
\[
= - \sum_{j \in \mathcal{N}_i} (x_i(t_j') - x_i(kh) - x_j(t_j') + x_j(kh)) + x_i(kh) - x_j(kh))
\]
\[
= - \sum_{j \in \mathcal{N}_i} (x_i(t_j') - x_i(kh)) + \sum_{j \in \mathcal{N}_i} (x_j(t_j') - x_j(kh)) - \sum_{j \in \mathcal{N}_i} (x_i(kh) - x_j(kh))
\]
\[
\dot{z}_i(t) = -Le(kh) + Lz(kh) = -\tilde{z}_i(t).
\]

Then, the dynamics of agent \( i \) for \( t \in [kh, (k + 1)h) \) can be given by
\[
\dot{x}_i(t) = - \sum_{j \in \mathcal{N}_i} (x_i(t_j') - x_j(t_j'))
\]
\[
= - \sum_{j \in \mathcal{N}_i} (x_i(t_j') - x_i(t,kh) - x_j(t_j') + x_j(t,kh))
\]
\[
= - \sum_{j \in \mathcal{N}_i} (x_i(t_j') - x_i(t,kh)) + \sum_{j \in \mathcal{N}_i} (x_j(t_j') - x_j(t,kh)) - \sum_{j \in \mathcal{N}_i} (x_i(t,kh) - x_j(t,kh))
\]
\[
= - \sum_{j \in \mathcal{N}_i} (x_i(t,kh) - x_j(t,kh)),
\]

which can be written in a compact form
\[
\dot{x}_i(t) = -Le(kh) + Lz(kh) = -\tilde{z}_i(t).
\]

Now we provide the main results.

**Theorem 1** Multi-agent systems (1) with a fixed, undirected, and connected communication topology \( G \) applying the consensus protocol (5) with the event condition (4) achieve the average consensus if and only if

\[
\dot{x}(t) = \frac{x^T(t)Lx(t)}{2}.
\]

**Proof** Take a candidate Lyapunov function \( V(t) = \frac{x^T(t)Lx(t)}{2} \). Combing the Taylor expansion of \( x(t) \) at \( t = kh \) and Equation (10), we can obtain for any \( t \in [kh, (k + 1)h) \)
\[
\dot{V}(t) = x^T(t)Lx(t)
\]
\[
= -x^T(t)Lx(t)
\]
\[
= -x^T(t)(Lx(t) + Lx(kh)) - x^T(t)Lx(t)
\]
\[
\leq -\tilde{z}_i(t)^2 + Lx(kh)^2 + \tilde{z}_i(t)Lx(t)
\]
\[
= e^T(t)\tilde{z}_i(t)^2(e^T(t)Lx(t)) + \tilde{z}_i(t)Lx(t).
\]

Considering \( e^T(t)L^2x(t) \leq \frac{\alpha}{2}e^T(kh)\tilde{z}_i(t)^2(e^T(t)L^2x(t)) + \frac{1}{2a}e^T(t)\tilde{z}_i(t)^2 + \tilde{z}_i(t)Lx(t) \) and the event condition (4), \( \dot{V}(t) \) can be bounded as
\[
\dot{V} \leq -\tilde{z}_i(t)^2 + \left( \frac{\alpha}{2}e^T(kh)\tilde{z}_i(t)^2(e^T(t)L^2x(t)) + \frac{1}{2a}e^T(t)\tilde{z}_i(t)^2 + \tilde{z}_i(t)Lx(t) \right)
\]
\[
\leq (-1 + \frac{1}{2a}h)\tilde{z}_i(t)^2 + \frac{\alpha e^T(kh)\tilde{z}_i(t)^2}{2}e^T(t)Lx(t)
\]
\[
\leq (1 - m_{\text{max}})(-1 + \frac{1}{2a}h)\tilde{z}_i(t)^2 + \frac{\alpha e^T(kh)\tilde{z}_i(t)^2}{2}e^T(t)Lx(t).
\]
where \( m_{\text{max}} = \max \{ m_i | i = 1, \ldots, N \} \). Noticing \( \ddot{x}(t) \dot{z}(t) \geq 0 \), thus we get that \( \ddot{V}(t) \) is negative semi-definite under the conditions (11). Because \( V(t) \geq 0 \), \( \dot{V}(t) \leq 0 \) implies that \( \ddot{V}(t) \) has a finite limit and \( \lim_{t \to \infty} \ddot{V}(t) = 0 \). At the same time, considering 
\[
0 = \lim_{t \to \infty} \ddot{V}(t) \leq (1 - m_{\text{max}})(-1 + \frac{2L}{\sqrt{2}} + h \lambda \dot{z}(t) \dot{z}(t) \leq 0 ,
\]
we have \( \lim_{t \to \infty} \dot{z}(t) = 0 \). It follows from Equation (4) that \( \lim_{t \to \infty} Lx(t) = 0 \). Then we obtain \( \lim_{t \to \infty} \dot{z}(t) = \lim_{t \to \infty} Lx(kh) + Le(kh) = 0 \). This further leads to \( \lim_{k \to \infty} Lx(kh) = 0 \). Since the considered graph \( G \) is fixed, undirected and connected, the Laplacian matrix \( L \) only has single zero eigenvalue with the corresponding eigenvector \( 1_x \). Therefore, \( \lim_{k \to \infty} x_j(kh) = \lim_{k \to \infty} x_j(kh) \quad i, j \in \{1, 2, \ldots, N\} \), which means consensus is achieved. Meanwhile, because \( \frac{1}{N} \sum_{i=1}^{N} x_i(t) \) is invariant, we further get Equation (6). Thus, the proof of Theorem 1 is completed.

**Remark 2** In this paper, each agent needs to be aware of partial global information, i.e., the largest eigenvalue of \( L \), to estimate \( h \) and \( \alpha \). In order to avoid this restraint, we find an upper bound on \( \lambda \), given by \( \lambda < 2\hat{\lambda}_{\text{max}} \leq 2(N-1) \) in Ref. [28]. Therefore, we can choose appropriate \( h \) and \( \alpha \) with constraint \( \frac{1}{\alpha} + 2h < \frac{1}{N-1} \) and use \( \left\| e_i(t) + \alpha h \right\| \geq \frac{\alpha}{\alpha} \left( \frac{1}{N-1} - \frac{1}{\alpha} - \frac{2h}{\alpha} \right) \left\| x_i(t) \right\| \) for replacing the event condition (4).

### 4. NUMERICAL SIMULATIONS

In this section, numerical simulations are presented to illustrate the effectiveness of the proposed event-based control strategy. Here we consider multi-agent systems with four agents whose dynamics are described by (1). The corresponding communication topology is shown in Figure 2.

![Network topology composed of four agents](image)

**Figure 2** Network topology composed of four agents

We can get \( \hat{\lambda} = 4 \). In order to make the superiorities of our algorithm more intuitive, we apply analogous event conditions and identical initial values \( x(0) = [0.4773, -0.3392, 0.5, -0.6381]^T \) as those in Ref. [24]. Furthermore, according to Equation (4), we choose the sampling period and the parameters of the event condition for each agent as \( h = 0.002 \), \( m_1 = m_2 = 0.55 \), \( m_3 = 0.33 \), \( m_4 = 0.99 \), and \( \alpha = 4.76 \), respectively. The state evolution of each agent is shown in Figure 3. It can be seen from Figure 3 that multi-agent systems (1) applying the consensus protocol (5) achieve the average consensus. Figure 4 and Figure 5 show the event instants for each agent, which is equipped with the event mechanism proposed in this paper and that in Ref. [24], respectively. It is apparent that our number of event instants is less than that in Ref. [24], which means fewer control updates happen during the same period.

![The state evolution of the systems (1) applying the protocol (5)](image)

**Figure 3** The state evolution of the systems (1) applying the protocol (5)
5. CONCLUSIONS

A novel event-triggered consensus control protocol for single-integrator multi-agent systems is proposed in this paper. The control updates and communication under the proposed protocol only happen at event times, resulting in significantly reducing the energy consumption. Employing the Lyapunov function method, the sufficient conditions have been obtained to guarantee the average consensus. It is worth noting that all agents are equipped with an identical sampling period, so one of future works is devoted to the case of heterogeneous sampling periods.

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