A Penalty/MPQI Method for Constrained Parabolic Optimal Control Problems

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Abstract
This paper is concerned with the numerical implementation and testing of penalty/MPQI algorithm for solving problems in PDE-constrained optimization. It uses this method to transform a constrained optimal control problem into a sequence of unconstrained optimal control problems. It is shown that the solutions to the original constrained problem. Convergence results are proved numerically.

Keywords: Optimal control, Exterior penalty function, parabolic Equations, MPQI technique, Distributed parameter systems.

1. Introduction and Statement of the Problem
This paper exposes a methodology (penalty/MPQI) allowing us to solve a constrained optimal control problem (COCP) for parabolic systems. This methodology belongs to the class of exterior penalty methods (EPMs). A penalty function approach commonly considered in finite dimensional optimization problem is employed. An augmented performance index is generally considered in penalty methods for both finite optimization problem and optimal control problem. It is constructed as the sum of the original cost function and so-called penalty functions that have some diverging asymptotic behavior when the constraints are approached by any tentative solution. This augmented performance index can then be optimized in the absence of constraints, yielding a biased estimate of the solution of the original problem. The weight of the penalty functions is gradually reduced to provide a converging sequence, hopefully diminishing the bias. The penalty function methods are computationally appealing, as they yield unconstrained problems for which a vast range of highly effective algorithms are available. In finite dimensional optimization, outstanding algorithms have resulted from the careful analysis of the choice of penalty functions and the sequence of weights. In this context, EPMs have been applied to optimal control problems by Winkler [1], Xing [2], Salim [3], Farag [4,5] and others [6-9].

In this study, we consider the following problem of minimizing the cost functional:

\[
J(v) = \int_0^T \left[ y(x, t; v) - f_0(x) \right]^2 \, dx + \alpha \int_0^T \left[ v(t) - \omega(t) \right]^2 \, dt
\]

subject to:
\[
\frac{\partial y}{\partial t} = \frac{\partial}{\partial x} \left( \lambda(x) \frac{\partial y}{\partial x} \right) + F(x, t), \quad (x, t) \in \Omega = [0, l] \times [0, T]
\]
(2)

\[y(x, 0) = \phi(x), \quad 0 \leq x \leq l\]
(3)

\[\frac{\partial y(x, t)}{\partial x} \bigg|_{x=0} = 0, \quad 0 < t \leq T\]
(4)

\[-\lambda(x) \frac{\partial y(x, t)}{\partial x} \bigg|_{x=l} = \eta \left[ G(y(l, t)) - \nu(t) \right], \quad 0 < t \leq T\]
(5)

\[\beta_0 \leq y(x, t) \leq \beta_1, \quad (x, t) \in [0, l] \times [0, T]\]
(6)

where \(\alpha, \beta_0, \beta_1, \lambda, \eta, \nu\) and \(T\) are positive numbers, \(F(x, t) \in L_2(\Omega), \phi(x) \in L_2[0, l], \lambda(x) \in L_\infty[0, l]\) are given functions. The functions \(\lambda(x), \nu\) are continuous and their derivatives are bounded. Here \(\omega(t)\) plays role of a guess for the
solutions or a selection for the solutions: as (1)-(6) could have many solutions to, we choose that which is nearest to \(\omega(t)\), by minimizing (1). It is proved in [10] that there is a unique weak solution to (2)-(6).

The problem (1)-(6) can be understood as a vibrational formulation of the inverse problem where one wishes to find the heating \(\omega(t)\) at a part of the boundary such that at the final time the temperature of the system reaches a given desired goal \(f_0(x)\). The particular case \(G(y) = y^4\) corresponding to Stefan-Boltzmann radiation condition seems to have many applications in engineering [11-12].

2. MODIFIED FUNCTIONAL AND ITS GRADIENT FORMULA

The constrained optimal control problem (1)-(6) is converted to an unconstrained control problem by adding a penalty function [13] to the cost functional (1), yielding the modified functional \(\Psi_{\alpha,k}(v,r_k)\) as follows

\[
\Psi_{\alpha,k}(v,r_k) \equiv \Psi(v) = f_\alpha(v) + P_k(v),
\]

where

\[
\Phi^1(y) = \left[ \max \left\{ \beta_0 - y(x,t); 0 \right\} \right]^T, \quad \Phi^2(y) = \left[ \max \left\{ y(x,t) - \beta_1; 0 \right\} \right]^T
\]

\[
P_k(v) = r_k \int_0^T \left[ \frac{\partial \mathcal{F}}{\partial y} + \frac{\partial \mathcal{F}}{\partial x} \right] dt
\]

and \(r_k > 0, k = 1, 2, \cdots\) are positive numbers and \(\lim_{k \to \infty} r_k = +\infty\).

The gradient of the cost functional with respect to the control variable can be efficiently calculated by means of the adjoint system. Let us now begin with the strong version of the Lagrangian, where we adjoin all constraints by separate multipliers \(\Theta(x,t), \Theta(0,t), \Theta(l,t), \Theta(x,0)\), and perform partial integrations with respect to space and time.

\[
L(y,\Theta,x,t) = \int_0^T \left[ y(x,T;v) - f_0(x) \right]^2 dt + \alpha \int_0^T \left[ v(t) - \omega(t) \right]^2 dt
\]

\[
+ \int_0^T \left[ \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} \left( \lambda(x) y \frac{\partial y}{\partial x} \right) + \lambda(x) \frac{\partial y}{\partial x} \right] dt + r \left[ \Phi^1(y) + \Phi^2(y) \right]
\]

\[
+ \int_0^T \left[ y(x,0) - \phi(x) \right] \Theta(x,0) dx + \int_0^T \left[ \frac{\partial y}{\partial t} \right] \Theta(x,0) dt
\]

\[
+ \int_0^T \left[ -\eta \left[ G(y(l,t)) - y(v(t)) \right] - \lambda(x) \frac{\partial y}{\partial x} \right] \Theta(l,t) dx
\]

(8)

The vibrational of the functional \(L(y(x,t),\Theta(x,t),x,t)\) is as follows

\[
\Delta L(y,\Theta,x,t) = 2 \int_0^T \left[ y(x,T;v) - f_0(x) \right] \Delta y(x,T;v) dx
\]

\[
+ 2 \alpha \int_0^T \left[ v(t) - \omega(t) \right] \Delta v(t) dt + \alpha \left[ \Delta v(t) \right]^2 dt
\]

\[
+ \int_0^T \left[ + \lambda(l) \frac{\partial \Theta(l,t)}{\partial x} + \eta \frac{\partial G(y(l,t))}{\partial y} \Theta(l,t) \right] \Delta y(l,t) dt
\]

\[
+ \int_0^T \left[ + \frac{\partial \Theta}{\partial t} + \frac{\partial y}{\partial x} \left( \lambda(x) \frac{\partial \Theta}{\partial x} \right) + r_k \left[ \frac{\partial \Phi^1(y)}{\partial y} + \frac{\partial \Phi^2(y)}{\partial y} \right] \right] \Delta y(x,t) dx dt
\]

\[
+ \int_0^T \left[ \Delta y(x,T;v) \right]^2 dx + \int_0^T \Theta(x,T) \Delta y(x,T) dx + \int_0^T \Theta(x,0) \Delta y(x,0) dx
\]

(9)
In (9), differentiation with respect to the state \( y(x,t) \) and additional partial integration with respect to space and time lead, by means of usual vibrational arguments, to the following adjoint system

\[
\frac{\partial \Theta}{\partial t} + \frac{\partial}{\partial x} \left( \lambda(x) \frac{\partial \Theta}{\partial x} \right) = -r_1 \left( \frac{\partial \Phi^1(y)}{\partial y} + \frac{\partial \Phi^2(y)}{\partial y} \right), \quad (x,t) \in \Omega \tag{10}
\]

\[
\Theta(x,T) = 2 \left[ y(x,T,v) - f_0(x) \right], \quad 0 \leq x \leq l \tag{11}
\]

\[
-\lambda(x) \left. \frac{\partial \Theta(x,t)}{\partial x} \right|_{x \rightarrow l} = \eta \left. \frac{\partial G(y(l,t))}{\partial y} \right|_{x \rightarrow l} \Theta(l,t), \quad 0 < t \leq T \tag{12}
\]

By helping the obtaining adjoint system (10)-(12), we are going to give the following theorem for computing the gradient of the modified functional \( \Psi_{a,k}(v, r_k) \).

**Theorem 1**

Let the above conditions of the optimal control problem (2)-(5) and (7) hold. Then the functional \( \Psi_{a,k}(v, r_k) \) is Frechet differentiable \( \frac{\partial \Psi_{a,k}(v,r_k)}{\partial v} \in L^2_2[0,T] \). Moreover, Frechet derivative at \( v(t) \) of the functional \( \frac{\partial \Psi_{a,k}(v,r_k)}{\partial v} \)

\[
\text{can be defined by the solution} \quad \Theta(x,t) \in W^{1,0}_2(\Omega) \text{ of the adjoint problem (10)-(12) as follows:}
\]

\[
\frac{\partial \Psi_{a,k}(v,r_k)}{\partial v} = \psi(v) = -\frac{\partial H(y(x,t),\Theta(x,t),v(t))}{\partial v} \tag{13}
\]

where \( H(y(x,t),\Theta(x,t),v(t)) \) is defined by

\[
H(y(x,t),\Theta(x,t),v(t)) \equiv -\left[ \alpha \left\{ v(t) - \omega(t) \right\}^2 + \eta \Theta(l,t) v(t) \right] \tag{14}
\]

**Proof:** The proof is similar to that of Theorem 5 [14].

### 3. Numerical Procedure

#### 3.1 PQI Technique

The partial quadratic interpolation method [15] or shortly (PQI) technique may be considered as a second-order method, the principal idea of this method is to approximate the objective function \( f(u') \) about certain point \( u' \) by second degree polynomial in the space \( R^p \), from which one may determine approximations to the gradient and Hessian of this function as \([b_n(u')]\) and \([A_n(u')]\) respectively. One then extract a positive definite matrix \([A_n(u')]\) from the Hessian matrix \([A_n(u')]\) in the subspace \( R^p \subset R^n \), using a particular application of the cholesky technique. The following flowchart of the Partial Quadratic Interpolation Technique is as follows in Figure 1 and \( x' = u' \).

#### 3.2 MPQI Technique

For Modified PQI [16], in PQI technique we set \( \theta = 1, \frac{1}{2}, \frac{1}{4}, \cdots \) and we take the first value of \( \theta \) which satisfies the condition \( f(u'^{\text{th}}) < f(u') \), \( u' \in R^p \). However, the value of \( \theta \) taken by this may not be the optimal value of
Since there is a great possibility that the optimal value $\theta^*$ lies between these values, i.e. between $\theta = \frac{1}{2}, \frac{1}{4}, \ldots$ to get the optimal step size $\theta^*$ we suggest the following modification:

Let us approximate $f(u) = f(u^* + \theta \delta u^*)$ by a polynomial of second degree $p_2(\theta)$ over the interval $[0,1]$ as following:

$$f(\theta) = p_2(\theta) = \left[ \frac{1}{h} \left( \frac{\theta}{h} \right)^2 \right] [L_2] \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

(15)

where $h$ is the interval of the interpolation, $f_{2i} = f(u^* + i \delta u^*)$, $i = 0, \frac{1}{2}, 1$ and $L_2$ is the Lagrange matrix where

$$L_2 = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

(16)

from (21) and (22) we have

$$p'(\theta) = \left[ \frac{1}{h} \left( \frac{\theta}{h} \right) \right] \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

(17)

then, $\frac{d p(\theta)}{d \theta} = \frac{1}{h} \left( -\frac{2}{3} f_0 + 2 f_1 - \frac{1}{2} f_2 \right) + \frac{2 \theta^*}{h^2} \left( \frac{1}{2} f_0 - 4 f_1 + f_2 \right)$ and taking $h = \frac{1}{2}$ we obtain

$$\theta^* = \frac{3 f_0 - 4 f_1 + f_2}{4 (f_0 - 2 f_1 + f_2)}$$

(18)

Figure 1: Partial Quadratic Interpolation Method
3.3 Iterative Algorithm (PMPQI) and its Convergence for COCP

The outlined of the algorithm (PMPQI) for solving the constrained optimal control problem (COCP) are as follows:

1) Given \( k = 0 \), \( \varepsilon^* > 0 \), \( r_k > 0 \), \( \varepsilon > 0 \) and \( v^{(k)} \in V \).

2) At each iteration \( k \) do

   Solve state system (2)-(4), then find \( y(.,v^{(k)}) \).

   Minimize \( \Psi(v^{(k)}) \) to find optimal control \( v^{(k+1)} \) using PMPQI technique.

   End do.

3) If \( \|\Psi(v^{(k+1)}) - \Psi(v^{(k)})\| < \varepsilon \), then Stop, else, go to Step 4.

4) Set \( v^{(k+1)} = v^{(k)} \), \( r_{k+1} = \varepsilon - r_k \), \( k = k + 1 \) and go to Step 2.

The following flowchart of the penalty function method combined with the Modified Partial Quadratic Interpolation Technique (Figure 2) is as follows:

![Flowchart](image)

**Figure 2:** Penalty Modified Partial Quadratic Interpolation Method

The following theorem represents the main contribution of the convergence theory for the control sequence, generated by the above numerical algorithm for the unconstrained optimal control problem.

**Theorem 2:**
Let \( \{v^k\} \) be a sequence of minimizers to the unconstrained optimal control problem problem which generated by the above numerical algorithm for any increasing sequence of values \( r_k \). Then \( \{v^k\} \) converge to the optimum solution \( v^* \) of the constrained optimal control problem as \( \lim_{k \to \infty} r_k = +\infty \).

**Proof:** The proof is similar to that of Theorem 4 [17].

4. **Numerical Results**

In this section we illustrate PMPQI technique for the optimal control problem (7),(2)-(4) with unconstrained (UOCP) and constrained (COCP) in the following example. Let \( \xi \) be a number in \((0,\pi]\). Then \( y(.,t) = e^{-\xi^2 t} \cos(\xi x) \) is the solution of the problem

\[
\frac{\partial y}{\partial t} = \lambda(x) \frac{\partial y}{\partial x}, \quad (x,t) \in [0,1] \times [0,T]
\]

(19)
\[ y(x,0) = \cos(\xi x), \quad 0 \leq x \leq l \]  
\[ \frac{\partial y(x,t)}{\partial x} \bigg|_{x=0} = 0, \quad -\lambda(x) \frac{\partial y(x,t)}{\partial x} \bigg|_{x=1} = -\xi e^{-\xi t} \sin(\xi), \quad 0 < t \leq T \]  
\[ \beta_0 \leq y(x,t) \leq \beta_1 \]  
\[ \frac{\partial y(x,t)}{\partial x} \bigg|_{x=1} = -y^4(l,t) + v(t) \]  
\[ v(t) = e^{-4\xi^2 t} \cos^4(\xi) - \xi e^{-\xi t} \sin(\xi) \]. We see that \[ v(t) = e^{-4\xi^2 t} \cos^4(\xi) - \xi e^{-\xi t} \sin(\xi) \]. Thus, in this example we set \[ G(y) = -y^4 \]. To test our method we set \[ f_0(x) = y(x,T) = e^{-\xi^2 T} \cos(\xi x) \]. In this example, \[ T \] is set to be 1 and the space and time grid sizes are taken to be 0.05 and 0.05, respectively. To analyze the effects of the initial guess on the reconstruction of the control function, the algorithm was run with three initial guess functions in figures 3, 4 for UOCP and COCP respectively. We tested our algorithm for \[ \xi = \pi \], then \[ v(t) = e^{-4\xi^2 t}, \quad f_0(x) = e^{-\pi^2 \xi^2} \cos(\pi x) \].

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**Figure 3:** Values of control function \( v(t) \) with 3 initial guesses (UOCP)

**Figure 4:** Values of control function \( v(t) \) with 3 initial guesses (COCP)
The numerical results are given in figures 5-9. In figure 5 shows the estimated control function compared with the exact one for UOCP and COCP. But in figures 6 and 7 show the values of the functional and gradient functional $\Psi_{\alpha,k}(v,r_k)$ various iterations numbers very close to one another and close to zero.

Figure 5: Values of control function $v(t)$ for UOCP, COCP and exact control

Figure 6: The values of function $\Psi_{\alpha,k}(v,r_k)$ various iteration numbers

Figure 7: The values of $\frac{\partial \Psi_{\alpha,k}(v,r_k)}{\partial v}$ various iteration numbers
But in Figures 8 and 9 show the comparison between the exact state results and the present numerical results for $y_{ij}^f$ in $x = i h$ and $t = j \tau$ when $h = \frac{1}{20}$, $\tau = \frac{1}{20}$ at $N = M = 8$ for UOCP and COCP. We note that the values of $y_{ij}^f$ for COCP problem converge to exact solution more accurate than UCOP problem.

![Figure 8](image_url)  
**Figure 8:** The values of state $y_{ij}^f$ at $i = j = 8$ for UOCP

![Figure 9](image_url)  
**Figure 9:** The values of state $y_{ij}^f$ at $i = j = 8$ for COCP

References


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